

# A DICHOTOMY FOR THE STABILITY OF ARITHMETIC PROGRESSIONS

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**ABSTRACT.** Let  $\mathcal{H}$  stand for the set of homeomorphisms  $\phi: [0, 1] \rightarrow [0, 1]$ . We prove the following dichotomy for Borel subsets  $A \subset [0, 1]$ :

- either there exists a homeomorphism  $\phi \in \mathcal{H}$  such that the image  $\phi(A)$  contains no 3-term arithmetic progressions;
- or, for every  $\phi \in \mathcal{H}$ , the image  $\phi(A)$  contains arithmetic progressions of arbitrary finite length.

In fact, we show that the first alternative holds if and only if the set  $A$  is meager (a countable union of nowhere dense sets).

## 1. DEFINITIONS

Let  $\mathbb{R}$ ,  $\mathbb{Q}$  denote the sets of real and rational numbers, respectively. By an AP (arithmetic progression) we mean a finite strictly increasing sequence in  $\mathbb{R}$  of the form  $\mathbf{x} = (x + kd)_{k=0}^{n-1}$ , with  $d > 0$  and  $n \geq 3$ . The convention is sometimes abused by identifying the sequence  $\mathbf{x}$  with the set of its elements. An AP is completely determined by its first term  $x = \min \mathbf{x}$ , its length  $n = |\mathbf{x}|$  and its step (difference)  $d > 0$ .

A subset  $S \subset \mathbb{R}$  is called FAP (free of APs) if it does not contain 3-term APs.

A subset  $S \subset \mathbb{R}$  is called RAP (rich in APs) if it contains APs of arbitrary large finite length.

Denote by  $\mathcal{H}$  the set of homeomorphisms  $\phi: [0, 1] \rightarrow [0, 1]$  of the unit interval. The result presented in the abstract can be restated as follows.

**Theorem 1.** *Let  $S \subset [0, 1]$  be a Borel subset. Then exactly one of the following two assertions holds:*

- (1) *(either) there exists a  $\phi \in \mathcal{H}$  such that  $\phi(S)$  is FAP;*
- (2) *(or)  $\phi(S)$  is RAP for every  $\phi \in \mathcal{H}$ .*

Moreover, (1) holds if and only if  $S$  is meager.

Recall some basic relevant definitions. Let  $S \subset \mathbb{R}$ . A set  $S$  is called *nowhere dense* if its closure  $\bar{S} \subset \mathbb{R}$  has empty interior.  $S$  is called *meager* (a set of first category), if it is a countable union of nowhere dense sets.  $S$  is called *residual*, or *co-meager*, if  $\mathbb{R} \setminus S$  is meager;  $S$  is called *residual* in a subinterval  $X \subseteq \mathbb{R}$  if the complement  $X \setminus S$  is meager. Finally,  $S$  is called *the set of second category* if it is not meager.

The following proposition lists some “largeness” properties of a set  $A \subset \mathbb{R}$  which force it to be RAP. Denote by  $\lambda$  the Lebesgue measure on  $\mathbb{R}$ .

**Proposition 1** (Classes of RAP sets). *Let  $A \subset \mathbb{R}$ . Then  $S$  is RAP if  $S$  belongs to at least one of the following four classes:*

$$\begin{aligned} \mathcal{E}_1 &= \{S \subset \mathbb{R} \mid S \text{ is Lebesgue measurable with } 0 < \lambda(S) \leq \infty\}, \\ \mathcal{E}_2 &= \{S \subset \mathbb{R} \mid S \text{ is residual in some interval } X \subset \mathbb{R} \text{ of positive length}\}, \\ \mathcal{E}_3 &= \{S \subset \mathbb{R} \mid S \text{ is winning in Schmidt's game}\}, \\ &\quad \text{(several versions of Schmidt's games are possible, see [5],[7]),} \\ \mathcal{E}_4 &= \{S \subset \mathbb{R} \mid S \text{ is Borel and not meager}\} \end{aligned}$$

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*Proof.* For a set  $S \in \mathcal{E}_1$ , one easily produces APs near any its Lebesgue density point. The argument for the classes  $E_2$  and  $E_3$  is even easier because residual subsets and the class  $E_3$  are closed under finite (and even countable) intersections.

Finally, the sets  $S \in \mathcal{E}_4$  are RAP because  $\mathcal{E}_4 \subset \mathcal{E}_2$ . (A Borel subset  $S \subset \mathbb{R}$  of second category must be residual in some subinterval, see e.g. Proposition 3.5.6 and Corollary 3.5.2 in [8, page 108]).  $\square$

Note that the the problems of finding finite or countable configurations  $F$  in sets  $S \subset \mathbb{R}$ , under various “largeness” metric assumptions on  $S$ , has been considered by several mathematicians.

Following Kolountzakis [3], a set  $F$  is called *universal* for a class  $\mathcal{E}$  of subsets of reals if  $F \ll S$  for all  $S \in \mathcal{E}$ . Henceforth  $F \ll S$  means that  $S$  contains an affine image of  $F$ , i.e. that  $aF + b \subset S$ , for some  $a, b \in \mathbb{R}$ ,  $a > 0$ . For example,  $S$  is RAP iff  $\{1, 2, \dots, n\} \ll S$  for all  $n \geq 1$ ;  $S$  is FAP iff  $\{1, 2, 3\} \ll S$ .

Every finite subset of reals is universal for all the classes  $\mathcal{E}_k$ ,  $1 \leq k \leq 4$ . Every bounded countable subset is universal for the classes  $\mathcal{E}_k$ ,  $2 \leq k \leq 4$ .

An old question of Erdős is whether there is an universal infinite set  $F \subset \mathbb{R}$  for the class  $\mathcal{E}_1$  (of sets of positive measure). The question is still open even though some families of countable sets  $F$  are shown not to contain universal functions, see Kolountzakis [3], Paul and Laczkovich [6] and references there. In [6] an elegant combinatorial characterization of universal sets  $F$  (for the class  $\mathcal{E}_1$ ) is given which reproduces earlier results in the subject.

Keleti [2] constructed a compact set  $A \subset [0, 1]$  of Hausdorff dimension 1 which is FAP; on the other hand, Laza and Pramanik in [4] showed that under certain assumptions (on the Fourier transform of supported measure) compact sets of fractional dimension close to 1 must contain 3-term APs (i.e., cannot be FAP). We refer to [4] for survey of related questions.

The central result of the paper, Theorem 1, completely characterizes the topological (rather than metric) properties of a Borel set  $S \subset \mathbb{R}$  which guarantee it to be RAP. This theorem is an immediate consequence of the following proposition and the fact that the sets  $S \in \mathcal{E}_4$  must be RAP (Proposition 1).

**Proposition 2.** *For every meager subset  $C \subset [0, 1]$ , there is a map  $\phi \in \mathcal{H}$ ,  $\phi: [0, 1] \rightarrow [0, 1]$ , such that  $\phi(C)$  is FAP.*

A stronger version of Proposition 2 (Proposition 3) is presented and proved in the next section.

## 2. PROOFS OF PROPOSITIONS 2 AND 3

Denote by  $\mathcal{C}$  the Banach space of continuous maps  $f: [0, 1] \rightarrow \mathbb{R}$  equipped with the norm

$$(2.1) \quad \|f\| = \|f\|_\infty = \max_{x \in [0, 1]} |f(x)|.$$

Denote by  $\mathcal{F}$  and  $\mathcal{H}^+$  the following subsets of  $\mathcal{C}$ :

$$(2.2) \quad \mathcal{F} = \{f \in \mathcal{C} \mid f \text{ is non-decreasing with } f(0) = 0; f(1) = 1\},$$

$$(2.3) \quad \mathcal{H}^+ = \{f \in \mathcal{F} \mid f \text{ is injective}\} = \{f \in \mathcal{H} \mid f \text{ is increasing on } [0, 1]\}.$$

The set  $\mathcal{F}$  is a closed subset of  $\mathcal{C}$ , while  $\mathcal{H}^+$  is residual in  $\mathcal{F}$ . (Indeed,

$$\mathcal{H}^+ = \bigcap_{\substack{0 < a < b < 1 \\ a, b \in \mathbb{Q}}} F_{a,b}; \quad F_{a,b} = \{f \in \mathcal{F} \mid f(a) < f(b)\}$$

where  $\mathbb{Q}$  stands for the set of rationals, and  $F_{a,b}$  are open dense subsets of  $\mathcal{F}$ ).

The following proposition is a stronger version of Proposition 2.

**Proposition 3.** *Let  $C \subset [0, 1]$  be a meager subset. Then, for residual subset of  $\phi \in \mathcal{H}^+$ , the image  $\phi(C)$  is FAP (has no 3-term APs).*

Since a meager set is a countable union of nowhere dense sets, it is enough to prove the above proposition under the weaker assumption that  $C$  is nowhere dense. Indeed, a meager set  $C$  has a representation in the form  $C = \bigcup_{i=k}^\infty C_k$  where  $C_k$  are nowhere dense. Then the unions  $U_k = \bigcup_{i=1}^k C_i$  form a nested sequence of nowhere dense sets, and  $\phi(C)$  is FAP if all  $\phi(U_k)$  are.

Let

$$(2.4) \quad \mathcal{H}_\varepsilon(C) = \{\phi \in \mathcal{H}^+ \mid \phi(C) \text{ has no 3-term APs of step } d \geq \varepsilon\}.$$

In the proof of Proposition 3 we need the following lemma. Its proof is provided in the end of the next section.

**Lemma 1.** *Let  $C \subset [0, 1]$  be a nowhere dense subset and  $\varepsilon > 0$ . Then  $\mathcal{H}_\varepsilon(C)$  contains a dense open subset of  $\mathcal{H}^+$ . In particular,  $\mathcal{H}_\varepsilon(C)$  is residual in  $\mathcal{H}^+$ .*

*Proof of Proposition 3 assuming Lemma 1.* We may assume that  $C$  is nowhere dense (see the sentence following Proposition 3). We may also assume that  $C$  is compact (otherwise replacing  $C$  by its closure  $\bar{C}$ ).

By Lemma 1, each of the sets  $\mathcal{H}_\varepsilon(C)$ ,  $\varepsilon > 0$ , is residual in  $\mathcal{H}^+$ . It follows that the set  $\mathcal{H}_0(C) = \bigcap_{k=1}^\infty \mathcal{H}_{1/k}(C)$  is residual. It is also clear that, for  $\phi \in \mathcal{H}_0(C)$ , the images  $\phi(C)$  are FAP.

This completes the proof of Proposition 3.  $\square$

### 3. PROOF OF LEMMA 1

First we prepare some auxiliary results.

**Lemma 2.** *Let  $C \subset [0, 1]$  be a nowhere dense set, let  $f \in \mathcal{H}^+$  and let  $\varepsilon > 0$  be given. Then there exists  $g \in \mathcal{H}^+$  such that  $\|g - f\| < \varepsilon$  and the set  $g(C)$  has no 3-term APs with step  $d \geq \varepsilon$ .*

*Proof.* Without loss of generality, we assume that  $\varepsilon < 1/2$ . Pick an integer  $r \geq 3$  such that  $r\varepsilon > 1$ .

Since  $C$  is nowhere dense, so is  $f(C)$ , and one can select  $r - 1$  points  $x_1, x_2, \dots, x_{r-1} \in (0, 1) \setminus f(\bar{C})$ ,

$$0 = x_0 < x_1 < x_2 < \dots < x_{r-1} < x_r = 1,$$

partitioning the unit interval into  $r$  subintervals  $X_k = (x_{k-1}, x_k)$ , each shorter than  $\varepsilon$ :

$$0 < |X_k| = x_{k+1} - x_k < \varepsilon \quad (1 \leq k \leq r).$$

Then one selects non-empty open subintervals  $Y_k = (y_k^-, y_k^+) \subset X_k$ ,  $1 \leq k \leq r$ , in such a way that the following four conditions are met:

$$(3.1) \quad \begin{aligned} (c1) \quad & f(C) \subset \bigcup_{k=1}^r \bar{Y}_k, \\ (c2) \quad & x_{k-1} < y_k^- < y_k^+ < x_k \text{ (i.e., } \bar{Y}_k \subset X_k), \text{ for } 2 \leq k \leq r-1, \\ (c3) \quad & 0 = x_0 = y_1^- < y_1^+ < x_1, \text{ and} \\ (c4) \quad & x_{r-1} < y_r^- < y_r^+ = x_r = 1. \end{aligned}$$

That is, between  $Y_j$  and  $Y_{j+1}$  there exists  $x_j \notin f(\bar{C})$  and  $|Y_j| < |X_j| < \varepsilon$  for all  $j$ .

Set  $p_1 = 0$ ,  $p_r = 1$  and then select the  $r - 2$  points  $p_k \in Y_k$ ,  $2 \leq k \leq r - 1$ , so that the set  $P = \{p_k\}_{k=1}^r$  contain no 3-term APs. Then the sequence  $(p_k)_1^r$  is strictly increasing, and

$$\delta = \min_{1 \leq m < n < k \leq r} |p_m + p_k - 2p_n| > 0.$$

Next, for  $1 \leq k \leq r$ , we select open subintervals  $Z_k \subset Y_k$ , each shorter than  $\frac{\delta}{4}$ , with  $p_k \in \bar{Z}_k$ .

Define  $u \in \mathcal{H}$  to be the homeomorphism  $[0, 1] \rightarrow [0, 1]$  which affinely contracts  $\bar{Y}_k$  to  $\bar{Z}_k$  and affinely expands the gaps between the intervals  $\bar{Y}_k$  to fill it in. Note that

$$(3.2) \quad |u(x) - x| < \varepsilon, \quad \text{for } x \in \bigcup_{k=1}^r \bar{Y}_k,$$

because  $x \in \bar{Y}_k$  implies  $u(x) \in \bar{Y}_k$  and hence  $|u(x) - x| \leq |Y_k| < |X_k| < \varepsilon$ .

Since  $u(x) - x$  is linear on each of the  $(r - 1)$  gaps between the intervals  $\bar{Y}_k$ , the inequality (3.2) extends to the whole unit interval:  $\|u(x) - x\| < \varepsilon$ .

Define  $g \in \mathcal{H}$  as the composition  $g(x) = (u \circ f)x = u(f(x))$ . Then

$$\|g - f\| = \|u \circ f - f\| = \|u(x) - x\| < \varepsilon.$$

It remains to show that  $g(C)$  has no 3-term APs with step  $d \geq \varepsilon$ . In view of (3.1),

$$\bigcup_{k=1}^r \bar{Z}_k = h\left(\bigcup_{k=1}^r \bar{Y}_k\right) \supset h(f(C)) = g(C),$$

so it would suffice to prove that  $\bigcup_{k=1}^r \bar{Z}_k$  has no 3-term APs with step  $d \geq \varepsilon$ .

Assume to the contrary that such an AP exists, say  $a_1, a_2, a_3$ , with  $d = a_2 - a_1 = a_3 - a_2 \geq \varepsilon$ . Let  $a_i \in \bar{Z}_{k_i}$ , for  $i = 1, 2, 3$ . These  $k_i$  are uniquely determined, and since  $|Z_{k_i}| < |X_{k_i}| < \varepsilon \leq d$ , we have  $k_1 < k_2 < k_3$ . Taking in account that  $|a_i - p_{k_i}| \leq |Z_{k_i}| < \delta/4$ , we obtain

$$\begin{aligned} |a_1 + a_3 - 2a_2| &\geq |p_{k_1} + p_{k_3} - 2p_{k_2}| - \\ &\quad - (|a_1 - p_{k_1}| + |a_3 - p_{k_3}| + 2|a_2 - p_{k_2}|) > \delta - 4 \cdot \frac{\delta}{4} = 0, \end{aligned}$$

a contradiction with the assumption that  $a_1, a_2, a_3$  forms an AP.  $\square$

**Corollary 1.** *Let  $C \subset [0, 1]$  be a nowhere dense set. Then for all  $\varepsilon > 0$ , the sets  $\mathcal{H}_\varepsilon(C)$  (defined by (2.4)) are dense in  $\mathcal{H}^+$ .*

*Proof.* Note that the sets  $\mathcal{H}_\varepsilon(C)$  are monotone in  $\varepsilon > 0$ :  $\mathcal{H}_{\varepsilon_2}(C) \subset \mathcal{H}_{\varepsilon_1}(C)$  if  $0 < \varepsilon_2 < \varepsilon_1$ .

By the previous lemma (Lemma 2), all sets  $\mathcal{H}_\varepsilon(C)$  are  $\varepsilon$ -dense. Then, for a given  $\varepsilon > 0$ , the set  $\mathcal{H}_\varepsilon(C)$  is  $\delta$ -dense for every positive  $\delta < \varepsilon$  (because even the smaller set  $\mathcal{H}_\delta(C) \subset \mathcal{H}_\varepsilon(C)$  is  $\delta$ -dense). This argument completes the proof of Corollary 1.  $\square$

**Lemma 3.** *Let  $C \subset [0, 1]$  be a compact nowhere dense set, let  $g \in \mathcal{H}$  and let  $\varepsilon > 0$  be given. Assume that the set  $g(C)$  has no 3-term APs with step  $d \geq \varepsilon$ . Then there exists a  $\delta > 0$  such that for all  $h \in \mathcal{H}$  such that  $\|h - g\| < \delta$  the sets  $h(C)$  have no 3-term APs with step exceeding  $2\varepsilon$ .*

*Proof.* Let

$$M = \{(x_1, x_2, x_3) \in g(C)^3 \mid x_2 - x_1 \geq \varepsilon \text{ and } x_3 - x_2 \geq \varepsilon\}.$$

Then  $M$  is compact, and  $F: M \rightarrow \mathbb{R}$  defined by  $F(x_1, x_2, x_3) = |x_1 + x_3 - 2x_2|$  assumes its minimum

$$\gamma = \min_{\mathbf{x} \in M} F(\mathbf{x}) > 0$$

which is positive because  $g(C)$  has no 3-term APs with step  $d \geq \varepsilon$ . Take  $\delta = \min(\varepsilon/2, \gamma/5)$ .

Assume to the contrary that for some  $h \in \mathcal{H}$  with  $\|h - g\| < \delta$ , the set  $h(C)$  contains an AP with step  $d' > 2\varepsilon$ , i.e. that there are  $c_1, c_2, c_3 \in C$  such that

$$h(c_3) - h(c_2) = h(c_2) - h(c_1) > 2\varepsilon.$$

Then, for both  $i = 1, 2$ , we have

$$g(c_{i+1}) - g(c_i) > h(c_{i+1}) - h(c_i) - 2\delta > 2\varepsilon - 2\delta \geq \varepsilon,$$

whence  $(g(c_1), g(c_2), g(c_3)) \in M$  and hence

$$\begin{aligned} \gamma &\leq F(g(c_1), g(c_2), g(c_3)) = |g(c_1) + g(c_3) - 2g(c_2)| \leq \\ &\leq |h(c_1) + h(c_3) - 2h(c_2)| + 4\delta = 0 + 4\delta \leq \frac{4\gamma}{5} < \gamma, \end{aligned}$$

a contradiction.  $\square$

*Proof of Lemma 1.* It follows from Lemma 3 that there is an (intermediate) open subset  $U \subset \mathcal{H}^+$  such that

$$\mathcal{H}_\varepsilon(C) \subset U \subset \mathcal{H}_{2\varepsilon}(C) \subset \mathcal{H}^+.$$

This set  $U$  is dense in  $\mathcal{H}^+$  because its subset  $\mathcal{H}_\varepsilon(C)$  is (by Corollary 1). Thus the set  $\mathcal{H}_{2\varepsilon}(C)$  contains an open dense subset  $U \subset \mathcal{H}^+$ . Since  $\varepsilon > 0$  is arbitrary, the proof is complete.  $\square$

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